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AN ANALYSIS OF VISCOELASTIC DAMPING CHARACTERISTICS OF A SIMPLY-SUPPORTED SANDWICH BEAM

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NOMENCLATURE

English Symbols		Units
x, y, z	rectangular coordinates	
t	time coordinate	sec.
a	length of the beam	in.
c	thickness of the core	in.
f	thickness of the lower facing	in.
f'	thickness of the upper facing	in.
E	Young's modulus of facing material	psi
E_{xz}	static elasticity modulus of the core material	psi
E_d	dynamic elasticity modulus of the core material	psi
G_{xz}	static shear modulus of the core material	psi
G_d	dynamic shear modulus of the core material	psi
u_c, w_c	displacements of the core in x and z directions respectively	
w	displacement of any point of the sandwich in z direction	
V_c	elastic energy per unit width of the core	in.-lb./in.
V_{MF}, V'_{MF}	elastic energy per unit width associated with membrane strains in the facings	in.-lb./in.
V_{BF}, V'_{BF}	elastic energy per unit width associated with bending strains in the facings	in.-lb./in.
V	total elastic energy per unit width of the beam	in.-lb./in.

English Symbols		Units
T	total kinetic energy per unit width of the beam	in.-lb./in.
I_T	$\frac{ff'}{f+f'}\left(c + \frac{f+f'}{2}\right)^2$	in.^{3}
I_F	$\frac{f^3+f'^3}{12}$	in.^{3}
s	$\frac{\pi^2 cff'}{a^2(f+f')}\ \frac{E}{G_{xz}(1-\nu^2)}$	
m	mode number	
A_m, B_m, C_m, F_m	configuration parameters	
E_1, E_2	total energy per unit width of the beam at instants 1 and 2	in.-lb./in.
L_{12}	total loss of energy per unit width of the beam over the period 1-2	in.-lb./in.

Greek Symbols		Units
α_m	exponential damping constant of the 'm'th mode	sec.$^{-1}$
ν	Poisson's ratio	
ρ	mass density per unit length of the beam	lb.-sec.^{2}/in.^{3}
δ	mass density per unit volume of the core material	lb.-sec.^{2}/in.
σ, τ	normal and shear stresses, respectively	psi
ε, γ	normal and shear strains, respectively	
σ_z, τ_{xz}	normal and shear stress, respectively, in the core	psi

Greek Symbols		Units
$\varepsilon_{zc}, \gamma_{xzc}$	normal and shear strains, respectively, in the core	
Φ	damping coefficient	lb.-sec./in.^{2}
Φ_m	damping coefficient of the 'm'th mode	lb.-sec./in.^{2}
$\omega_m, \bar{\omega}_m$	undamped and damped natural frequency, respectively, of the 'm'th mode	sec.$^{-1}$
Δ	logarithmic decrement	

SUMMARY

It has long been observed that a number of organic compounds such as rubbers, plastics, resins etc., possess an outstanding property of dissipating energy of motion to heat energy. The dual elastic and viscous properties exhibited by these materials together add up to render such dissipative characteristics.

In a simple sandwich beam the energy dissipation is brought about by shear-stresses induced in the constrained viscoelastic core by the two facings. Though various attempts have been made for the analysis of a simple sandwich beam, none of them is simple enough to be interpreted directly for design purposes.

This investigation is an extension of the work done by Kimel et al. (1)*, who analyzed the undamped natural frequency of a simply-supported sandwich beam. This extension leads to a simple closed form expression for damping of the principal mode of vibration. The accuracy of this simple result was verified by means of experiments performed with beams of different core-facing combination. The analytical values of logarithmic decrement were calculated by means of a Burroughs 5500 digital computer, making use of some of the material properties determined experimentally by Pearce (2). Though this analysis is principally limited to the first mode, which is

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*Numbers in brackets refer to the Literature Cited at the end.

the most critical mode from design considerations, it can very well be modified to take higher modes into consideration. The effect of rotary inertia was neglected in consideration of its small effect at low frequencies, but the effect of Poisson's ratio was taken into consideration. Moreover this simple analysis takes into account unequal facings, which very few other analyses do.

Once the material properties are known, this closed form expression for damping can be very effectively used for all design purposes in determination of the damping in the fundamental mode.

CHAPTER I

INTRODUCTION

Background

A critical factor in the design of dynamic systems is the control of vibrations. When any member of a system is exposed to random forces with broad-band characteristics, several resonances are excited simultaneously. However, in the majority of the cases, the fundamental natural frequency is the most critical design criterion as the amplitude of vibration often exceeds the factor of safety limit, resulting in failure of the member.

A careful review of the causes of vibration would reveal that it is absolutely impossible to eliminate vibration totally from any dynamic system. The best alternative is to endeavour to damp it out before any serious damage is inflicted on any member.

The most fundamental and effective mode of damping is through the use of external dampers. These, however cannot be used everywhere and moreover they increase the weight of the system considerably. External friction between vibrating parts may bring about a form of damping commonly known as 'Coulomb' damping. At times through more effective use of joints damping can be considerably increased. A form of stress-dependent damping inherent in every vibrating member, commonly termed as 'Solid' damping, has been very thoroughly analyzed by Lazan (3) and many others. A previous analysis by the author (4) revealed that in order to deploy this characteristic effectively, high concentration of stresses must be present. This is created by use of holes, slots, notches, and other stress raisers. The reduction of weight resulting from notching is not critical enough to be used effectively in low-weight systems.

Sandwich structures have aroused considerable interest in the post-war era. The remarkable ability of these low-weight structures to damp out vibrations rapidly has been a critical factor towards the rapid development of high speed aircrafts and missiles. A simple sandwich principally consists of a low density core sandwiched between two thin layers of metal facings. Variations in the structure of core attribute to the different forms of sandwich structures. A core in the form of a honeycomb - hexagonal cells perpendicular to the faces - makes up the 'Honeycomb' sandwich structure. In another variation, the core is in the form of corrugated sheets, with corrugations running parallel to the faces.

For all practical purposes a simple sandwich consisting of a plain viscoelastic core is the most versatile of all sandwich constructions. The major advantages of sandwich structures over conventional structures are:

1. Their overall density is low.
2. Their vibration damping characteristic is far more superior.
3. They have good structural rigidity.
4. They have good fatigue properties.

5. They provide good thermal and accoustical insulation.

6. They can be very easily mass produced.

Such overall versatility has contributed greatly to the development and mass production of both simple and complex sandwich structures during the last decade.

Review of Literature

In the recent years, a popular approach towards the analysis of sandwich constructions has been the Plantema (5) approach, based on energy principles. The first chapter of this book very vividly outlines the nature of deformations a vibrating sandwich beam undergoes. The author first attempted the analysis of damped vibration following Plantema approach, but found that it becomes too complicated especially with unequal facings and moreover this approach neglects the effect of Poisson's ratio, which though small, has some contribution. The present approach does however have a great deal of similarity to Plantema approach.

Jones et al. (6) analyzed the damping of a free-free sandwich beam with a shear-flexible core and they obtained large discrepancies in calculated and test values of amplitude decay which they explained was due to improper end conditions in their experimental set-up. The effect of rotary inertia, though negligible at lower modes, may have considerable effect at higher modes.

DiTaranto (7) has rigorously analyzed the free vibration of a three-layer beam through a complicated sixth-order differential equation. The integrals of these equations (8) resulting in highly complicated expressions even after restricting the analysis to sandwich beams having very thin viscoelastic layer as compared to the elastic layer. Needless to mention that for relatively thick layers of the core, his approach would be too cumbersome to be used effectively for design purposes. Derby and Ruzicka (9) however extended his theory further for the analysis of some structures and came up with fairly good agreement between theory and experimental results.

Objectives and Scope

The main objective of this investigation is to define a simple closed form expression for the damping produced by a viscoelastic core of a three layer simply-supported sandwich beam, which should be accurate and simple enough to be used effectively for design purposes. Various attempts have been made by previous investigators like Ungar and Ross (10), DiTaranto (7), and many others to obtain a closed form expression for predicting the damping in a three-layer beam but all have come up with expressions which can hardly be interpreted readily for design purposes. Moreover many of them have limited the applicability of their theory by assumptions like small damping and equal facing thickness.

The present analysis, the simplest of all those the author has come across, is more adaptable in the sense that it is not restricted by either of the two aforementioned assumptions. Even though this study is restricted to the analysis of the fundamental mode, which is the most critical mode from design considerations, the theory can very well be extended to take into account the higher modes. It is felt that the present analysis offers a new approach toward the analysis of damped vibration of sandwich structures which might prove to be much more versatile so far as its adaptability in design is concerned.

CHAPTER II

VISCOELASTIC MODELS

In classical theory of elasticity the basic assumption is that the stress-strain relations are linear and independent of time. However, a number of materials like plastics, resins, adhesives and rubbers have in addition a time dependent property, which attributes to their being termed as viscoelastic materials.

The structure and characteristics of viscoelastic materials have been very thoroughly studied by Williams (11), Flugge (12), Kolsky (13) and many others. The dynamic behavior of viscoelastic materials in uni-axial stress very closely resemble that of models built from discrete elastic and viscous elements.

There are two basic models from which any number of others may be built up through incorporation of additional spring and dashpot elements. Whether a given material performs according to one or another of these patterns is a question to be resolved by testing. The two basic forms are the Maxwell or fluid model (Figure 1) and the Kelvin or Voigt or solid model (Figure 2).

From Figure 1.

$$\varepsilon = \varepsilon_{spring} + \varepsilon_{dashpot}$$

$$\sigma_s = E_s \varepsilon_s \; ; \; \sigma_d = \Phi \frac{\partial \varepsilon_d}{\partial t} \; ; \quad \sigma = \sigma_s = \sigma_d$$

VISCOELASTIC MODELS

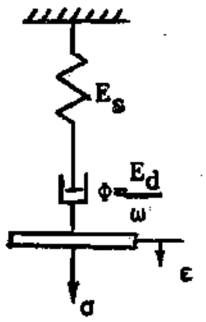


Figure 1. Maxwell Model

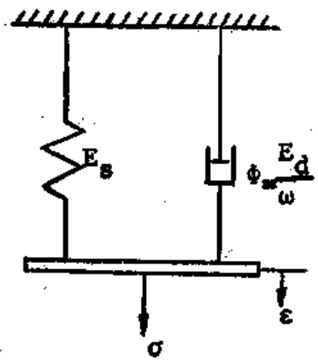


Figure 2. Voigt or Kelvin Model

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial \varepsilon_s}{\partial t} + \frac{\partial \varepsilon_d}{\partial t} = \frac{1}{E_s}\frac{\partial \sigma}{\partial t} + \frac{1}{\Phi} \cdot \sigma \tag{1}$$

Thus, $$\sigma = \Phi \frac{\partial \varepsilon}{\partial t} - \frac{\Phi}{E_s} \cdot \frac{\partial \sigma}{\partial t}$$

From Figure 2.

$$\sigma = \sigma_{spring} + \sigma_{dashpot}$$

$$\sigma_s = E_s \varepsilon_s \qquad \sigma_d = \Phi \frac{\partial \varepsilon_d}{\partial t} \tag{2}$$

$$\varepsilon = \varepsilon_s = \varepsilon_d$$

Hence, $$\sigma = E_s \varepsilon + \Phi \frac{\partial \varepsilon}{\partial t}$$

Within the realm of linearity of equation (1) the material shows a typical property of a fluid: its ability to undergo unlimited deformation under finite stress. The materials described by equation (1) are therefore termed as Maxwell fluids.

Equation (2) interprets the behavior of almost an elastic solid, the only difference being that the strain does not at once assume the final value, but approaches it gradually (delayed elasticity). The materials represented by equation (2) are therefore termed as Kelvin solids or Voigt solids.

It has been found that the spring constant E_s and the damping coefficient Φ of Voigt solids are not constants but are functions of frequency and temperature. More complex models should be used for exact analysis of viscoelastic materials but Pearce (2) showed that using Voigt model in conjunction with a modulus vs. frequency curve for constant temperature analysis is equivalent to using a more complex model.

In the present work, a Voigt model in conjunction with modulus-frequency curves as determined experimentally by Pearce (2) has been used for the constant temperature analysis of viscoelastic materials.

It should be noted that using a Voigt model is equivalent to using a complex modulus, E_c, i.e.

$$\sigma_c = E_c \varepsilon_c = (E_s + i\ E_d)\ \varepsilon \tag{3}$$

Let the system of Figure 2 be given a sinusoidal motion

$$\varepsilon = \varepsilon_0 \sin\omega t$$

then,

$$\dot{\varepsilon} = \omega\varepsilon_0 \cos\omega t = i\omega\varepsilon$$

Substituting this in equation (2) results:

$$\sigma = (E_s + i\omega\Phi)\ \varepsilon \tag{4}$$

It is apparent that equations (3) and (4) are identical, so that,

$$\Phi = \frac{E_d}{\omega} \tag{5}$$

From elementary theory of elasticity, the relationship between Elastic and Shear modulus is:

$$G = \frac{E}{2(1 + \nu)} \tag{6}$$

where ν is the Poisson's ratio of the material.

When the viscoelastic material undergoes shear deformation,

$$\phi = \frac{G_d}{\omega} \tag{7}$$

where

$$G_d = \frac{E_d}{2(1 + \nu_c)}$$

in which ν_c is the Poisson's ratio of the viscoelastic material.

CHAPTER III

THEORETICAL ANALYSIS

For a simply-supported sandwich beam, as shown in Figure 3, summation of forces in x and z directions yield

$$\frac{\partial \tau xz}{\partial z} = 0 \tag{8}$$

$$\frac{\partial \sigma_z}{\partial z} - \frac{\partial \tau_{xz}}{\partial x} = \delta \frac{\partial^2 w_c}{\partial t^2} \quad . \tag{9}$$

Consistent with the assumption that the core is infinitely rigid in transverse compression τ_{xz} and w_c can be expressed as

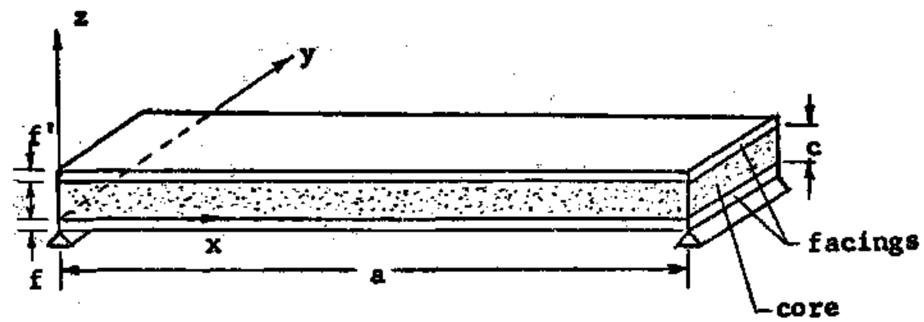
$$\tau_{xz} = \sum_{m=1}^{\infty} A_m \cos \frac{m\pi x}{a} \sin\omega_m t \tag{10}$$

$$w_c = \sum_{m=1}^{\infty} C_m \sin \frac{m\pi x}{a} \sin\omega_m t \quad . \tag{11}$$

Satisfying the boundary conditions for a simply-supported sandwich beam, one finds for all time t,

$$w_c(x = 0,a) = 0$$

a. Simply-Supported Beam



b. Differential Element of the Core

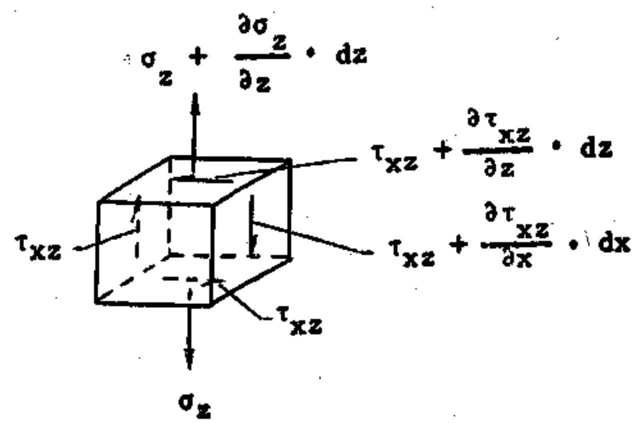


Figure 3.

and
$$\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x} - \gamma_{xzc}\right] (x = 0,a) = 0$$

where $\gamma_{xzc} = \tau_{xz}/G_{xz}$. The natural frequency of vibration has been determined by Kimel et al. (1) as

$$\omega_m^2 = \frac{m^4\pi^4 E}{a^4(1-\nu^2)} \left(\frac{I_T}{1+m^2 s} + I_F\right) \tag{12}$$

where

$$I_T = \frac{f\ f'}{f+f'}\ (c + \frac{f+f'}{2})^2$$

$$I_F = \frac{f^3+f'^3}{12}$$

and
$$s = \frac{\pi^2 c f f'}{a^2(f+f')} \cdot \frac{E}{G_{xz}\ (1-\nu^2)}$$

The equations (34), (35), and (36) of Kimel et al. (1) can be simplified to the forms

$$\left[\frac{a^2}{m^2\pi^2}\ \frac{1-\nu^2}{Ef'} + \frac{c}{G_{xz}}\right] A_m - \frac{m\pi}{a}\ (c + \frac{f'}{2})\ C_m + F_m = 0 \tag{13}$$

$$-(c+\frac{f'}{2})\ \frac{cf'}{G_{xz}} A_m + \frac{m\pi}{a} \left[\frac{f^3+f'^3}{3} + cf'(c+f') - \frac{a^4}{m^4\pi^4} \cdot \right.$$

$$\left. \frac{(1-\nu^2)}{E}\ \rho\omega_m^2\right] C_m + \left[\frac{f^2-f'^2}{2} - f'c\right] F_m = 0 \tag{14}$$

$$\frac{cf''}{G_{xz}} A_m + \frac{m\pi}{a} \left[\frac{f^2 - f'^2}{2} - f'c\right] C_m + (f + f') F_m = 0 \tag{15}$$

Equations (13) and (15) upon simplification yield

$$\frac{A_m}{C_m} = \frac{m^3 \pi^3}{a^3} \frac{E}{(1-\nu^2)(c + \frac{f+f'}{2})} \frac{I_T}{1+m^2 s} \tag{16}$$

For a non-conservative system, we may write

$$E_1 = E_2 + L_{12}$$

where E_1 and E_2 denote the total energy (kinetic and potential) of the system at instants 1 and 2, and L_{12} is the energy dissipated over this period.

The behavior of viscoelastic materials is best described by a Voigt model, as discussed earlier. For a single degree of freedom system describing the simple Voigt model, the equation of motion is

$$M \ddot{X} + C \dot{x} + K x = 0$$

whence,

$$x = e^{-\zeta\omega_n t} (C_1 \cos\omega_d t + C_2 \sin\omega_d t) \tag{17}$$

where $\omega_n = \sqrt{K/M}$, $\zeta = \dfrac{c}{2\sqrt{K\,M}}$, and $\omega_d^2 = \omega_n^2(1-\zeta^2)$

It has been shown by Kolsky (13) that for the analysis of any form of damped vibration, these simple equations can be modified by introducing the unknown exponential damping factor α, which is dependent upon the nature of damping being produced by the system. Thus for the sandwich beam, taking damping into consideration, the equation (11) can be modified as

$$w_c = \sum_{m=1}^{\infty} C_m e^{-\alpha_m t} \sin \frac{m\pi x}{a} \sin \bar{\omega}_m t \tag{18}$$

where $\bar{\omega} = \omega_d$, and $\alpha = \zeta\omega_n$, so that

$$\bar{\omega}_m^2 = \omega_m^2 - \alpha_m^2 \tag{19}$$

With this new deflection equation, all the Kinetic and Potential energy expressions have to be modified to take the damping into account. All the potential energy expressions of Kimel et al. (1), equations (11), (17), (18), (24) and (25) would be expressed as:

Strain energy in the core

$$V_c = \frac{ac}{4G_{xz}} \sum_{m=1}^{\infty} A_m^2 e^{-2\alpha_m t} \sin^2 \bar{\omega}_m t \tag{20}$$

Elastic strain energies in the facings associated with membrane strain in the facings

$$V_{MF} = \frac{Eaf'}{4(1-\nu^2)} \sum_{m=1}^{\infty} \left(\frac{m\pi}{a} F_m + \frac{f}{2} \frac{m^2\pi^2}{a^2} C_m \right)^2 e^{-2\alpha_m t} \sin^2 \bar{\omega}_m t \tag{21}$$

$$V_{MF}' = \frac{Eaf'}{4(1-\nu^2)} \sum_{m=1}^{\infty} \left[-\frac{A_m}{G_{xz}} \frac{m\pi c}{a} \quad \frac{m^2\pi^2}{a^2} \left(c + \frac{f'}{2} \right) C_m - \frac{m\pi}{a} F_m \right]^2 e^{-2\alpha_m t} \sin^2\bar{\omega}_m t \tag{22}$$

Elastic strain energy of the facings associated with bending of the facings about their own middle surfaces

$$V_{BF} = \frac{Eaf^3}{48(1-\nu^2)} \sum_{m=1}^{\infty} \frac{m^4\pi^4}{a^4} C_m^2 \; e^{-2\alpha_m t} \sin^2\bar{\omega}_m t \tag{23}$$

$$V_{BF}' = \frac{Eaf'^3}{48(1-\nu^2)} \sum_{m=1}^{\infty} \frac{m^4\pi^4}{a^4} C_m^2 \; e^{-2\alpha_m t} \sin^2\bar{\omega}_m t \tag{24}$$

The total elastic energy of the beam can be expressed as

$$\begin{aligned} V &= V_c + V_{MF} + V_{MF}' + V_{BF} + V_{BF}' \\ &= \sum_{m=1}^{\infty} \frac{ac}{4G_{xz}} \Bigg[A_m^2 + \frac{Eaf}{4(1-\nu^2)} \left(\frac{m\pi}{a} F_m + \frac{f}{2} \frac{m^2\pi^2}{a^2} C_m \right)^2 + \\ &\quad \frac{Eaf'}{4(1-\nu^2)} \left\{ -\frac{A_m}{G_{xz}} \frac{m\pi c}{a} + \frac{m^2\pi^2}{a^2} \left(c + \frac{f'}{2} \right) C_m - \frac{m\pi}{a} F_m \right\}^2 + \\ &\quad \frac{Ea(f^3 + f'^3)}{48(1-\nu^2)} \frac{m^4\pi^4}{a^4} C_m^2 \Bigg] e^{-2\alpha_m t} \sin^2\bar{\omega}_m t \end{aligned} \tag{25}$$

The total kinetic energy of the beam is

$$T = \frac{1}{4}\rho a \sum_{m=1}^{\infty} C_m^2(\bar{\omega}_m \cos\bar{\omega}_m t - \alpha_m \sin\bar{\omega}_m t)^2 \, e^{-2\alpha_m t} \tag{26}$$

For an element of unit width, the energy associated with the dashpot, which is dissipated by the core is given by

$$\frac{d^2L}{dx\,dz} = \int_{\gamma_1}^{\gamma_2} \tau d\gamma = \int_{\gamma_1}^{\gamma_2} \phi_m \dot{\gamma}_{xz} d\gamma \qquad \text{where } \phi_m = \frac{G_d}{\bar{\omega}_m}$$

$$\text{so that} \qquad \frac{d^2L}{dx\,dz} = \int_{t_1}^{t_2} \phi_m \dot{\gamma}_{xz}^2 dt = \int_{t_1}^{t_2} \frac{G_d}{\bar{\omega}_m} \dot{\gamma}_{xz}^2 dt \tag{27}$$

The core shear strain γ_{xzc} as evaluated in equation (9) of Kimel et al. (1), after modification for damping stands as

$$\gamma_{xzc} = \sum_{m=1}^{\infty} \frac{A_m}{G_{xz}} \, e^{-\alpha_m t} \cos\frac{m\pi x}{a} \sin\bar{\omega}_m t$$

from which

$$\dot{\gamma}_{xzc} = \sum_{m=1}^{\infty} \frac{A_m}{G_{xz}} (\bar{\omega}_m \cos\bar{\omega}_m t - \alpha_m \sin\bar{\omega}_m t) \; e^{-\alpha_m t} \cos\frac{m\pi x}{a} \tag{28}$$

Upon substitution of this expression, equation (27) takes the forms

$$\frac{d^2L_{12}}{dx\ dz} = \int_{t_1}^{t_2} \sum_{m=1}^{\infty} \frac{G_d}{\bar{\omega}_m} \frac{A_m^2}{G_{xz}^2} (\bar{\omega}_m \cos\bar{\omega}_m t - \alpha_m \sin\bar{\omega}_m t)^2\, e^{-2\alpha_m t} \cos \frac{m\pi x}{a}\, dt$$

$$= \sum_{m=1}^{\infty} \frac{G_d}{\bar{\omega}_m} \frac{A_m^2}{G_{xz}^2} \left[\frac{\bar{\omega}_m^2}{2} \left\{ -\frac{e^{-2\alpha_m t}}{2\alpha_m} + \frac{(\bar{\omega}_m \sin 2\bar{\omega}_m t - \alpha_m \cos 2\bar{\omega}_m t)}{2(\bar{\omega}_m^2 + \alpha_m^2)} e^{-2\alpha_m t} \right\} + \right.$$

$$\alpha_m \bar{\omega}_m \frac{(\alpha_m \sin 2\bar{\omega}_m t + \bar{\omega}_m \cos 2\bar{\omega}_m t)}{2(\bar{\omega}_m^2 + \alpha_m^2)} e^{-2\alpha_m t} - \frac{\alpha_m^2}{2} \left\{ \frac{e^{-2\alpha_m t}}{2\alpha_m} + \right.$$

$$\left. \left. \frac{(\bar{\omega}_m \sin 2\bar{\omega}_m t - \alpha_m \cos 2\bar{\omega}_m t)}{2(\bar{\omega}_m^2 + \alpha_m^2)} e^{-2\alpha_m t} \right\} \right]_{t_1}^{t_2} \cos^2 \frac{m\pi x}{a}$$

$$= \sum_{m=1}^{\infty} \frac{G_d}{\bar{\omega}_m} \frac{A_m}{G_{xz}^2} \left[e^{-2\alpha_m t} \left\{ -\frac{\alpha_m^2 + \bar{\omega}_m^2}{4\alpha_m} + \frac{\bar{\omega}_m^2 - \alpha_m^2}{4(\bar{\omega}_m^2 + \alpha_m^2)} \right. \right.$$

$$(\bar{\omega}_m \sin 2\bar{\omega}_m t - \alpha_m \cos 2\bar{\omega}_m t) + \frac{\alpha_m \bar{\omega}_m}{2(\bar{\omega}_m^2 + \alpha_m^2)}$$

$$\left. \left. (\alpha_m \sin 2\bar{\omega}_m t + \bar{\omega}_m \cos 2\bar{\omega}_m t) \right\} \right]_{t_1}^{t_2} \cos^2 \frac{m\pi x}{a}$$

Integrating over the volume and simplifying through the use of equation (19), the loss of energy

$$L_{12} = \frac{acG_d}{2G_{xz}^2} \sum_{m=1}^{\infty} \frac{A_m^2}{\bar{\omega}_m} \left[e^{-2\alpha_m t} \left\{ \frac{\bar{\omega}_m^2 - \alpha_m^2}{4\omega_m^2} (\bar{\omega}_m \sin 2\bar{\omega}_m t - \alpha_m \cos 2\bar{\omega}_m t) \right.\right.$$

$$\left.\left. + \frac{\alpha_m \bar{\omega}_m}{2\omega_m^2} (\alpha_m \sin 2\bar{\omega}_m t + \bar{\omega}_m \cos 2\bar{\omega}_m t) - \frac{\omega_m^2}{4\alpha_m} \right\} \right]_{t_1}^{t_2} \tag{29}$$

For convenience t_1 and t_2 are chosen as 0 and $2\pi/\bar{\omega}_m$ respectively. Any other choice of t_1 and t_2 would lead to the same solution for damping after much laborous simplification. Substituting the above chosen values of t_1 and t_2 in equation (29), the loss of energy at the end of first cycle is

$$L_{12} = \frac{acG_d}{2G_{xz}^2} \sum_{m=1}^{\infty} \frac{A_m^2}{\bar{\omega}_m} \left[- \frac{\bar{\omega}_m^2 - \alpha_m^2}{4\omega_m^2} \alpha_m + \frac{\alpha_m \bar{\omega}_m^2}{2\omega_m^2} - \frac{\omega_m^2}{4\alpha_m} \right] \left(e^{-4\pi\alpha_m/\bar{\omega}_m} - 1 \right)$$

$$= \frac{acG_d}{8G_{xz}^2} \sum_{m=1}^{\infty} \frac{A_m^2}{\alpha_m} \bar{\omega}_m \left(1 - e^{-4\pi\alpha_m/\bar{\omega}_m} \right) \tag{30}$$

Substituting the same values of t_1 and t_2 in the energy equations (25) and (26) and equating them to equation (30) by the relation

$$L_{12} = E_0 - E_{2\pi/\bar{\omega}_m}$$

the following equation results

$$\frac{acG_d}{8G_{xz}^2} \sum_{m=1}^{\infty} \frac{A_m^2}{\alpha_m} \bar{\omega}_m \left(1 - e^{-4\pi\alpha_m/\bar{\omega}_m}\right)$$

$$= \frac{1}{4} \rho a \sum_{m=1}^{\infty} c_m^2 \bar{\omega}_m^2 \left(1 - e^{-4\pi\alpha_m/\bar{\omega}_m}\right) \tag{31}$$

For the principal mode of vibration the mode number $m = 1$ and the above equation simplifies to yield the value of exponential damping constant

$$\alpha_1 = \frac{1}{2} \frac{c}{\bar{\omega}_1 \rho} \frac{G_d}{G_{xz}^2} \frac{A_1^2}{C_1^2} \tag{32}$$

Combining with equation (16), this yields

$$\alpha_1 = \frac{\pi^6}{2a^6} \frac{c}{\bar{\omega}_1 \rho} \frac{G_d}{G_{xz}^2} \frac{E^2}{(1-\nu^2)^2 (c + \frac{f+f'}{2})^2} \quad \frac{I_T^2}{(1+s)^2} \tag{33}$$

CHAPTER IV

EXPERIMENTAL INVESTIGATION

In order to verify the validity of the theory, a number of experiments were performed with different combinations of facing and core materials. All the tests were carried out at constant room temperature of 70°F, to eliminate the effect of temperature variation.

The models used were composed of steel and aluminum facings with Buna-N rubber (butadiene-acrylonitrile copolymer) and styrofoam as core materials. Epoxy cement (Sears, Roebuck and Co.) made up with a combination of a resin and a hardener was used as the bonding material. The surface of both facings and the core materials were cleaned thoroughly before being glued together, insuring proper adhesion. Curing was done by allowing the cement at least forty hours to dry at room temperature. A thin coating of cement was applied very uniformly so that the thickness of the bonding material was negligible as compared to the thickness of either facings or the core.

Instrumentation and Equipments

The various equipments used for the experiments are listed in the following table. A schematic diagram of the experimental set-up is shown in Figure 4.

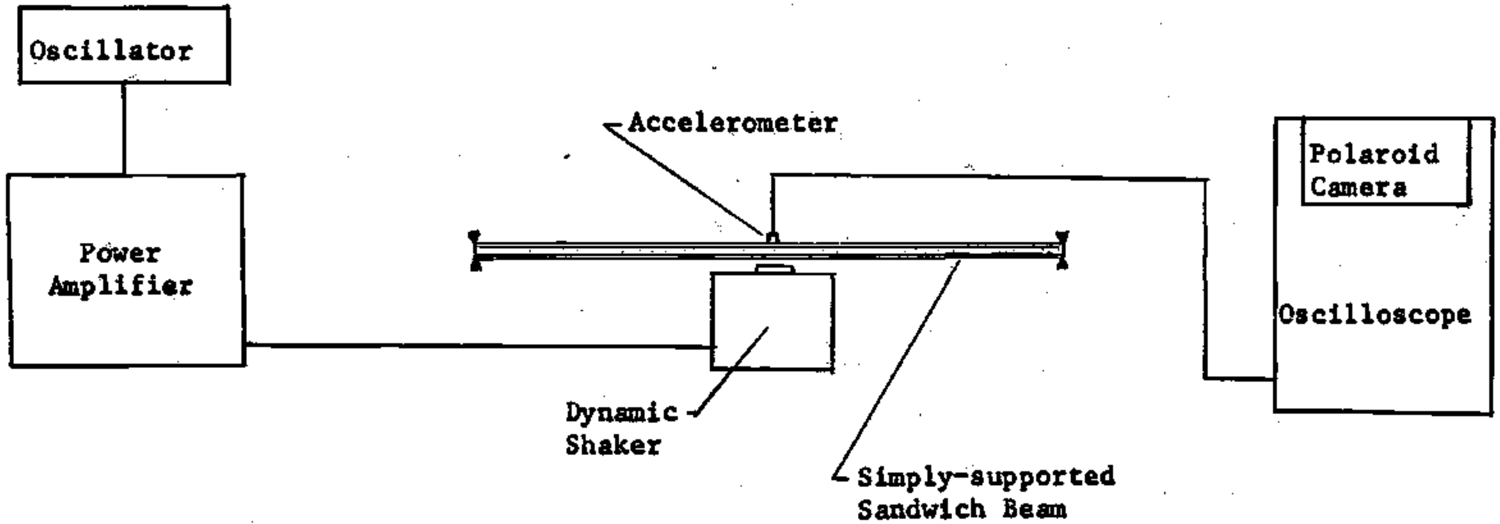


Figure 4. Schematic Diagram of Experimental Set-Up

Table 1. Experimental Equipments

Item	Manufacturer	Specifications
Audio Oscillator	Hewlett-Packard	Model 200CD
Power Amplifier	MB Electronics	Model 2125MB
Electrodynamic Shaker	MB Electronics	Model PM25
Accelerometer	Endevco Corp.	Model 2115E
Oscilloscope	Tektronix	Dual Beam Type 502A
Oscilloscope Camera	Tektronix	Polaroid Type C-12 Polaroid Type C-27

Experimental Procedure

The beam was simply-supported at the two ends on two rigid end supports with adjustable pressure mechanism to insure proper end conditions. The beam was vibrated by means of the electro-dynamic shaker placed below the beam, at the beam center, with a small air gap in between. The vibration frequency was raised to a value very close to the fundamental frequency of the beam so as to get maximum amplitude. A piezo-electric type accelerometer was fixed to the beam at its center by means of wax to pick up the vibration, and the signal was fed directly to an oscilloscope. The oscilloscope was set at single-sweep mode and the vibration was cut off suddenly by switching the power amplifier to standby mode. The resulting trace of damped vibration on the oscilloscope was recorded on a photograph by means of a Polaroid camera mounted on the oscilloscope. The experimental value of logarithmic decrement was calculated very carefully through accurate measurement of the peak amplitudes of the first decay cycle. The length of each beam, starting from 24 inches, was reduced by 2 inches successively and the same procedure was repeated every time until the beam was reduced to 12 inches in length. Different combinations of core and facing materials made up the different beams, all of which were tested by the same above mentioned procedure.

Material Properties

In order to calculate the natural frequency and logarithmic decrement for a particular beam, a number of material properties must be known. The common properties of Steel and Aluminum, taken from handbooks, are tabulated below.

Table 2. Metal Properties

Metal	Young's Modulus E (p.s.i.)	Specific Weight (lbs./in.^{3})	Poisson's Ratio ν
Steel	30×10^6	0.284	0.293
Aluminum	10.6×10^6	0.098	0.334

For the viscoelastic core materials, the densities were determined by direct measurement and the Poisson's ratio values were taken directly from Table 6 of Pearce's (2) thesis.

Table 3. Viscoelastic-Material Properties

Material	Specific Weight (lbs./in.^{3})	Poisson's Ratio ν
Buna-N Rubber (1/4")	0.042	0.50
Styrofoam	0.00098	0.155

The characteristic curves of static and dynamic moduli of elasticity vs. frequency for rubber and styrofoam were determined experimentally by Pearce (2) following the method laid down by Preiss and Skinner (14). These characteristic curves, as shown in Appendix A, were used for the analytical determination of damping.

CHAPTER V

RESULTS

The theoretically predicted values of logarithmic decrement obtained by solving equations (32) and (33) through use of the computer are now compared to the experimentally determined values. Tables 4 through 7 list the analytical values of $\bar{\bar{\omega}}$, the damped natural frequency, obtained from the computer solution. The reader is referred to Pearce's (2) work for a comparison of analytical and experimental $\bar{\bar{\omega}}$ for unconstrained layer structures.

Table 4. Steel and Rubber Composite

$f = f' = 0.031$ ins.; $c = 0.25$ ins.; $\rho = 0.000241$ lb-sec^2/in.^{3}

Beam Length a (ins.)	Analytical Damped Natural Frequency(c.p.s.)	Analytical Δ	Experimental Δ
24	21.448	0.334762	0.316670
22	24.154	0.356013	0.326684
20	27.442	0.375298	0.358945
18	31.516	0.393303	0.380055
16	36.661	0.415620	0.394654
14	43.299	0.441612	0.435318
12	52.118	0.466598	0.441833

Table 5. Steel and Styrofoam Composite

$f = f' = 0.031$ ins.; $c = 0.5$ ins.; $\rho = 0.0000831$ lb-sec^2/in.^{3}

Beam Length a (ins.)	Analytical Damped Natural Frequency(c.p.s.)	Analytical Δ	Experimental Δ
24	50.649	0.032991	0.033336
22	56.069	0.034228	0.031416
20	62.530	0.035428	0.037041
18	70.314	0.036648	0.038466
16	80.011	0.037752	0.040822
14	92.331	0.038823	0.040822
12	108.708	0.039740	0.042260

Table 6. Aluminum and Styrofoam Composite

$f = f' = 0.033$ ins.; $c = 0.375$ ins.; $\rho = 0.0000402$ lb-sec^2/in.^{3}

Beam Length a (ins.)	Analytical Damped Natural Frequency(c.p.s.)	Analytical Δ	Experimental Δ
24	52.952	0.022285	0.022473
22	59.916	0.024265	0.026682
20	68.325	0.026337	0.029559
18	78.612	0.028510	0.029414
16	91.407	0.030785	0.034486
14	107.636	0.033189	0.037458
12	129.215	0.035415	0.040822

Table 7. Aluminum and Rubber Composite

$f = f' = 0.033$ ins.; $c = 0.25$ ins.; $\rho = 0.0001385$ lb-sec^2/in.^{3}			
Beam Length a (ins.)	Analytical Damped Natural Frequency(c.p.s.)	Analytical Δ	Experimental Δ
24	22.635	0.211622	0.211309
22	26.011	0.233259	0.220241
20	30.178	0.256434	0.216233
18	35.386	0.285141	0.271934
16	42.111	0.315924	0.325422
14	50.901	0.354160	0.356675
12	62.917	0.390053	0.358945

The experimental and analytical values of logarithmic decrement are plotted against frequency for the above four combinations in Figures 5 through 8.

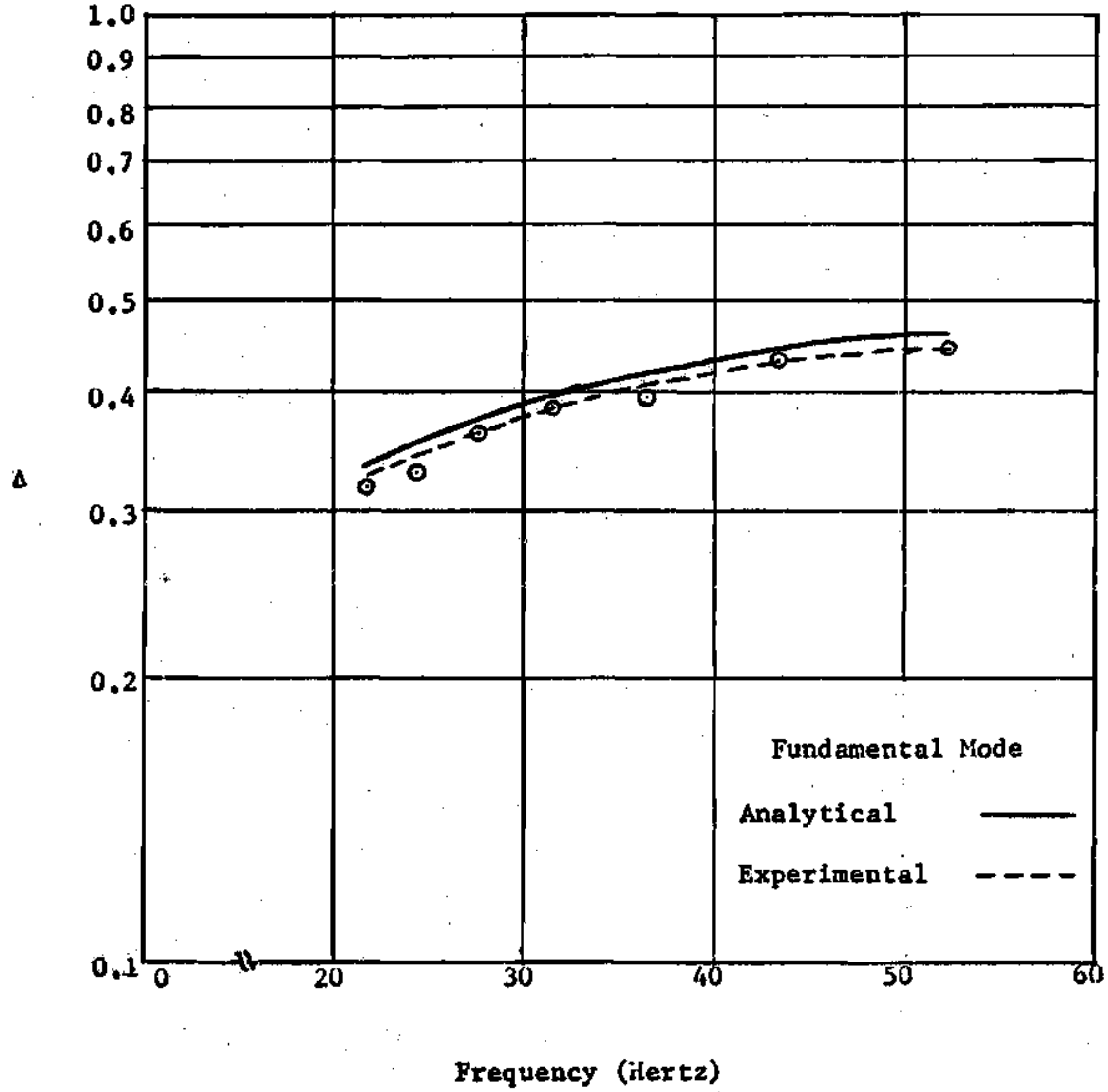


Figure 5. Steel and Rubber Composite

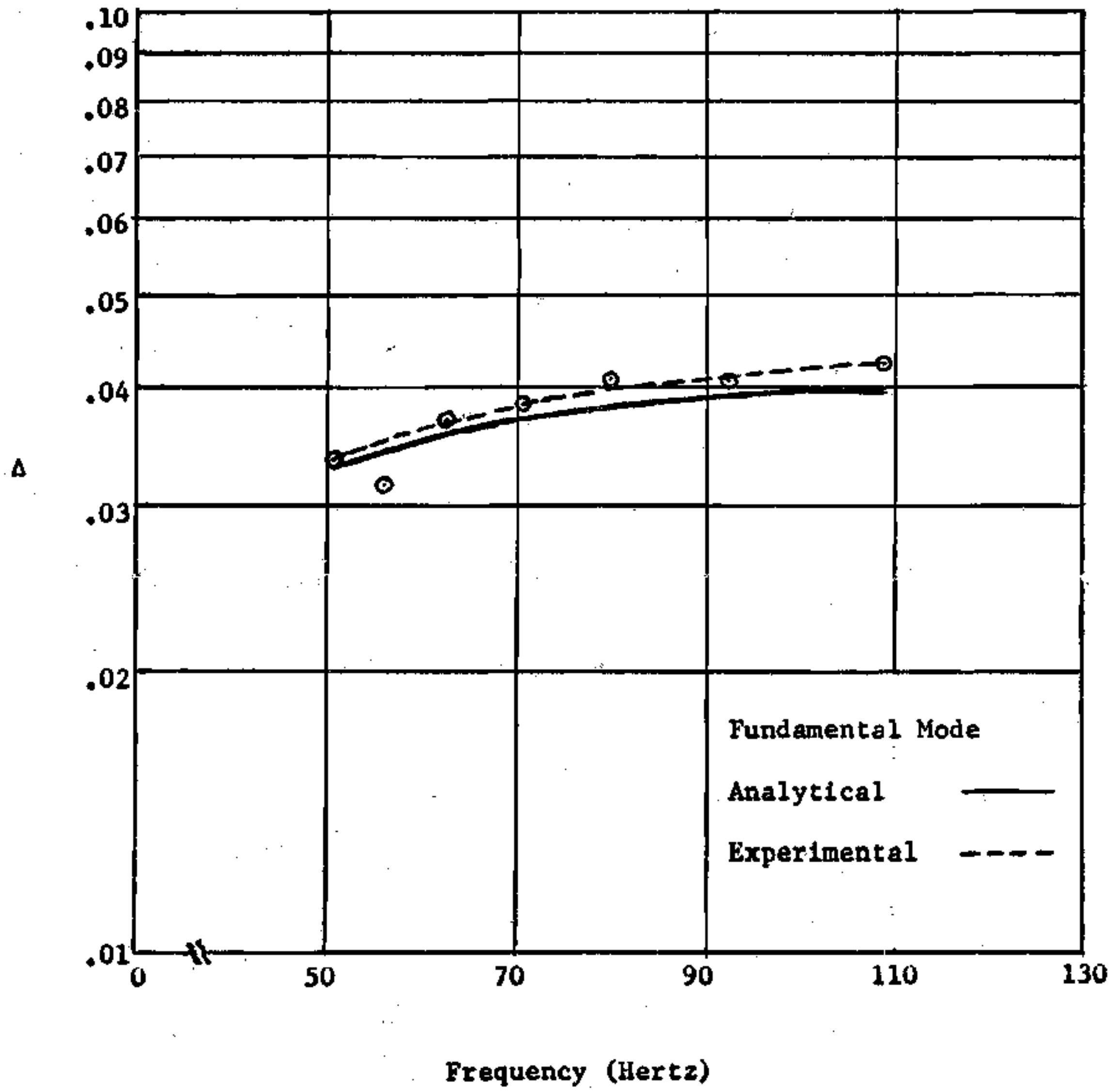


Figure 6. Steel and Styrofoam Composite

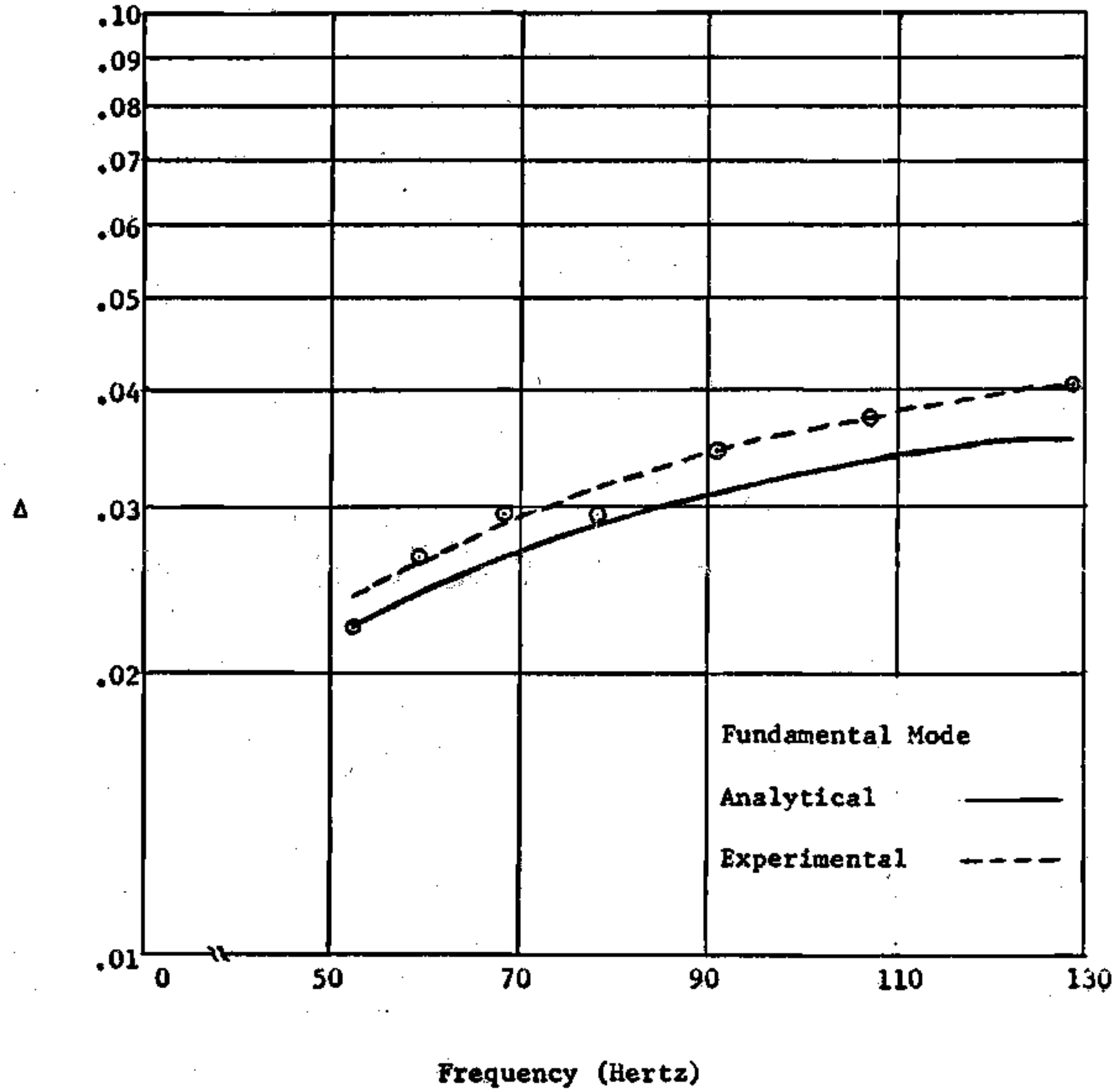


Figure 7. Aluminum and Stryofoam Composite

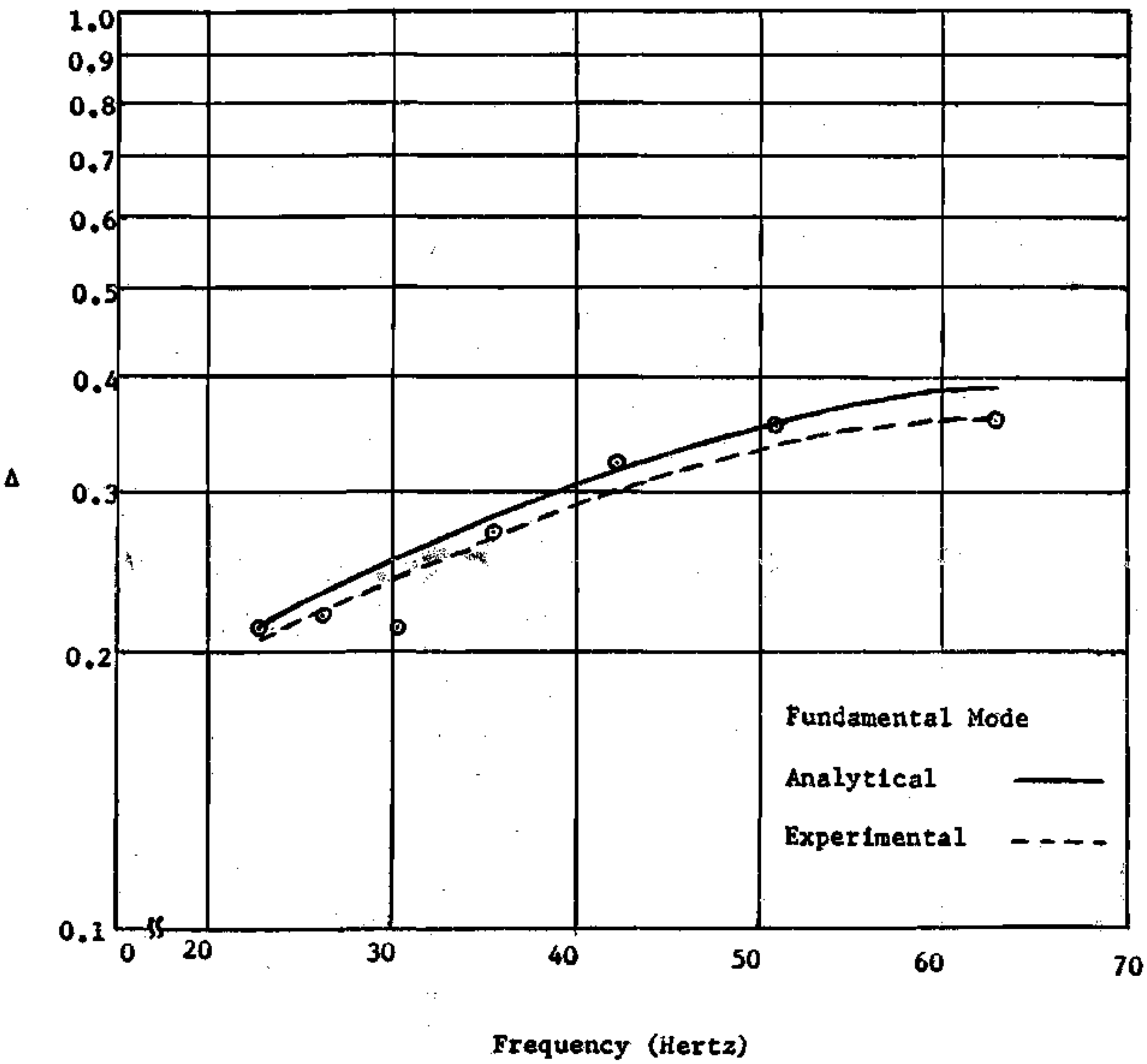


Figure 8. Aluminum and Rubber Composite

CHAPTER VI

CONCLUSIONS

A direct comparison between the theoretically predicted and the experimentally determined values of damping lead to the conclusion that the present analysis is accurate enough to be used very effectively for all design purposes. Studying Figures 5 through 8, one can conclude that the accuracy of the analysis tends to diminish with increasing frequency. This can very easily be explained by the observation that the effect of rotary inertia, which has been neglected in the present analysis, becomes predominantly important at higher frequencies.

It has been shown through dimensional analysis in Appendix D that damping is totally independent of the overall density of the beam. It is further observed that damping is dependent principally on the static and dynamic moduli characteristics of the core material and if these remain constant with respect to frequency, damping would be affected very little by changes in length, core thickness, and width of facings.

The present analysis can very easily be extended to analyze the higher modes. For any discrete higher mode, the subscripts 1 in Equation (32) would be replaced by 'm' of that particular mode. It is once again observed that damping is dependent on the frequency - moduli characteristic of the core material alone for any particular beam and is consequently independent of the mode number. The analysis, however, has some limitations at higher modes since it neglects the effect of rotary inertia, which becomes increasingly predominant at higher modes.

The present analysis is, as it stands, a complete work. The work can be extended to take into account the effect of rotary inertia to interpret the higher modes of vibration more accurately. Further extensions of this work can be the analysis of beams with different end conditions. Such an analysis would principally be effective for the determination of the damped natural frequency upon which the moduli values of core materials are dependent. The actual value of damping is, however, independent of end conditions.

The author feels that the present analysis has opened up a new approach to the treatment of sandwich beams and would be very effective for the design of all kinds of sandwich structures.

APPENDIX A

VISCOELASTIC MATERIAL CHARACTERISTICS

The dynamic and static characteristics of Buna-N rubber and styrofoam, as determined experimentally by Pearce (2), are shown in Figures 9 and 10. To determine these properties, he followed the method established by Preiss and Skinner (14). Equation (6) was used to convert the elasticity modulus values to shear modulus.

To determine analytically the values of logarithmic decrement, a trial and error procedure was followed, in which after determining the frequency of vibration correctly, the corresponding dynamic shear modulus values were fed in to get the final analytical values of Δ_A

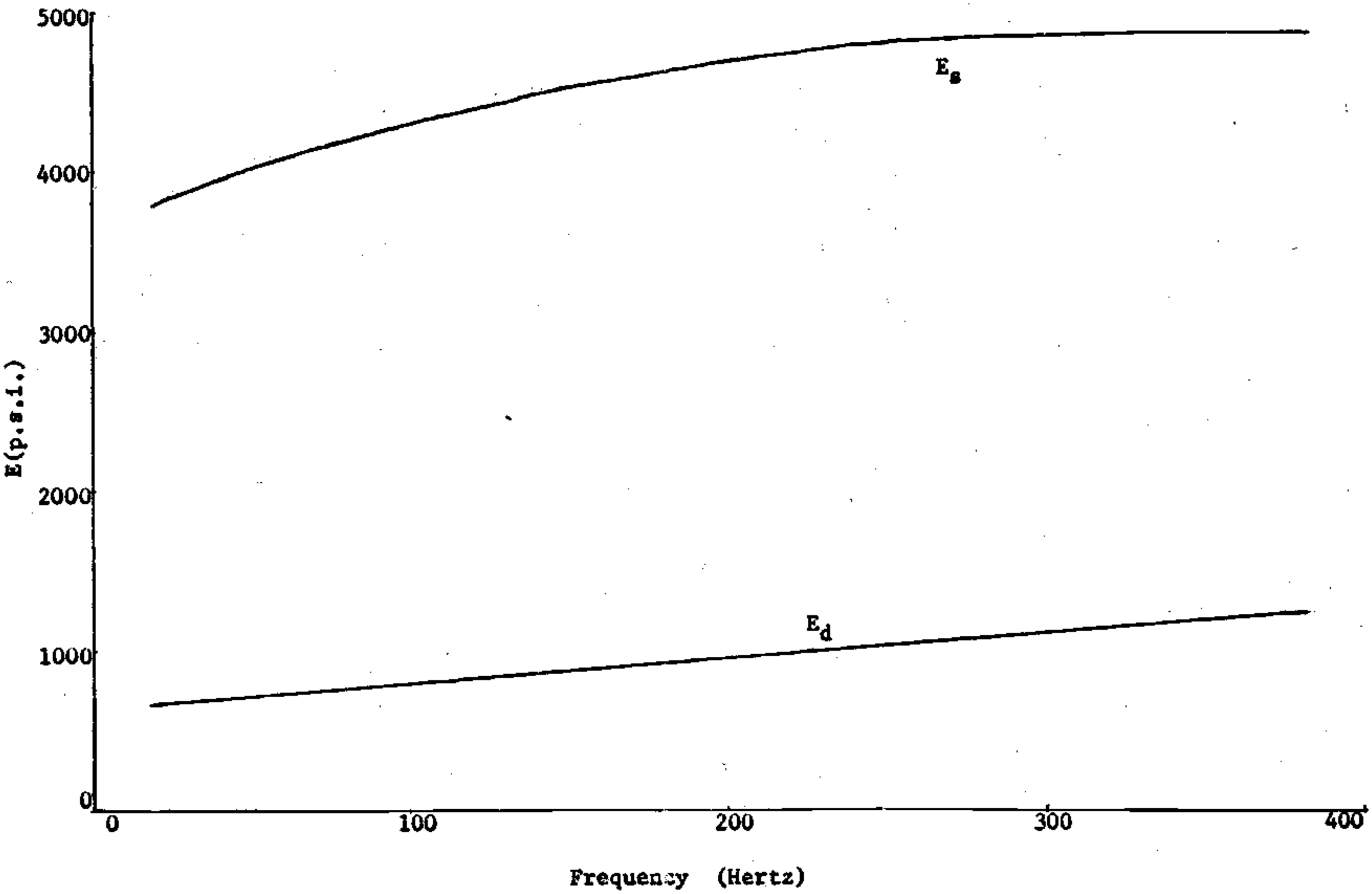


Figure 9. Static and Dynamic Moduli of 1/4" Buna-N Rubber

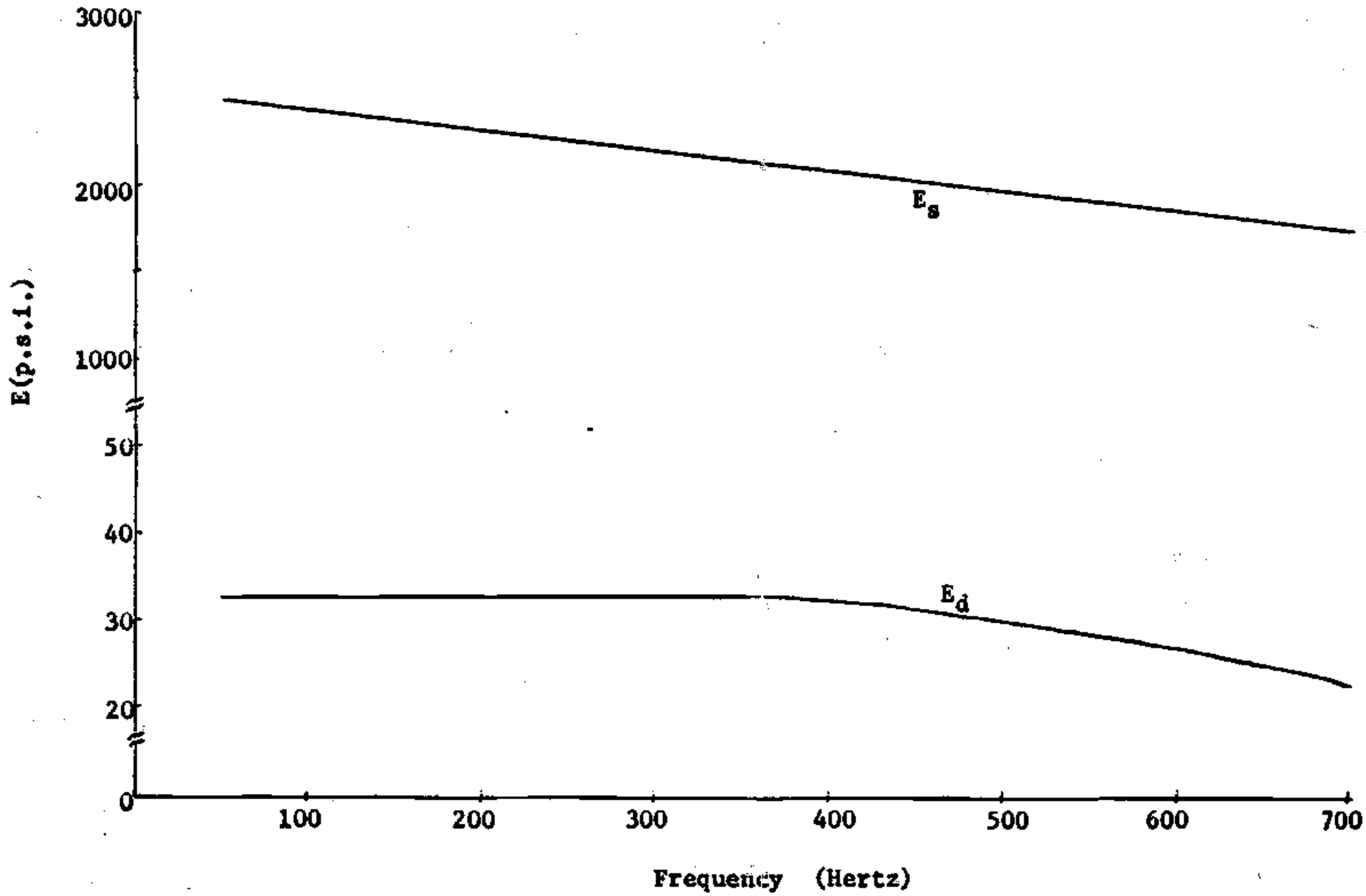


Figure 10. Static and Dynamic Moduli of Styrofoam

APPENDIX B

COMPUTER PROGRAM

(Burroughs B-5500 Computer)

ALGOL

```
BEGIN

FILE IN ACI1 (2,10); FILE OUT ACO1 16(2,15);

INTEGER E,G,A;

REAL NU,RHO,DELTA,A1,W,W1,C,F,F1,I,IT,S,GD,PROD,FN;

FORMAT FT1(X21,"IT=",F11.6,//,X22,"I=",F11.6,//,X22

       "S=",F11.6,//,X22,"W=",F11.6);

FORMAT FT2(X10,"ALPHA x W BAR=",F11.6);

FORMAT FT3(/,X18,"W BAR=",F11.6);

FORMAT FT4(/,X18,"ALPHA=",F11.6,//,X18,"DELTA=",F11.6);

FORMAT FT5(X6,"NATURAL FREQUENCY=",F11.6," CYCLES PER SECOND",///);

FORMAT FT6(X12,"DATA VALUES:",//,X12,"BEAM LENGTH=",I4,

       " INCHES",/,X22,"F=",F8.3," INCHES",/,X21,

       "F1=",F8.3," INCHES",/,X9,"CORE THICKNESS=",

       F8.3," INCHES",/,X21,"NU=",F8.3,/,X20,"RHO=",

       F12.7,/,X22,"E=",I8,/,X22,"G=",I4,/,X21,"GD=",F8.3,//);

READ(ACI1,/,A,F,F1,C,NU,RHO,E,G,DG);

WRITE(ACO1,FT6,A,F,F1,C,NU,RHO,E,G,GD);

       IT←FxF1/(F+F1)x(C+(F+F1)/2)*2;
```

```
        I←(F*3+F1*3)/12;
        S←(3.1416*2xCxFxF1)/(A*2x(F+F1))X
            E/(Gx(1-NU*2));
        W←SQRT ((3.1416*4xE)/(A*4xRHOx(1-NU*2))x
            (IT/(1+S)+I));
WRITE(AC01,FT1,IT,I,S,W);
        FN←W/(2x3.1416);WRITE(AC01,FT5,FN);
        PROD←(3.1416*6xCxGD/(2xA*6xRHOxG*2))x((Ex
            IT)*2/((1-NU*2)x(C+(F+F1)/2)x(1+S))*2);
WRITE(AC01,FT2,PROD);
        W1←SQRT ((W*2+SQRT (W*4-4xPROD*2))/2);
WRITE(AC01,FT3,W1);
        A1←PROD/W1;DELTA←(2x3.1416xA1*2)/PROD;
WRITE(AC01,FT4,A1,DELTA);
END.
```

Data for this program must be fed in on one card, each data value being separated by a comma (free-feed form). The data values should be punched in the following order: a, f, f', c, ν, ρ, E, G_{xz}, G_d, - the calculated values of I_T, I, s, ω, $\alpha_1 x \bar{\omega}_1$, $\bar{\omega}_1$, α_1, and Δ would be printed out.

APPENDIX C

DAMPING RELATIONS

Relation Between α and Δ

By definition,

$$\Delta = \ln\left(\frac{X_n}{X_{n+1}}\right) = \ln\left(\frac{X_o e^{-\alpha t_n}}{X_o e^{-\alpha(t_n+T)}}\right)$$

where, X_o - initial amplitude

X_n - amplitude at any instant, t

X_{n+1} - amplitude after one cycle

and T - time period $2\pi/\bar{\omega}$

Therefore,

$$\Delta = \ln\ (e^{\alpha T}) = \alpha T = \frac{2\pi\alpha}{\bar{\omega}}$$

APPENDIX D

DIMENSIONAL ANALYSIS

To indicate how the different variables affect the value of damping, a dimensional analysis is needed to relate Δ with all the other variables.

From equation (19), for the first mode

$$\bar{\omega}_1^2 = \omega_1^2 - \alpha_1^2$$

whence,

$$\bar{\omega}_1^4 - \bar{\omega}_1^2\omega_1^2 + (\alpha_1\bar{\omega}_1)^2 = 0$$

so that

$$\bar{\omega}_1^2 = \frac{\omega_1^2 + \sqrt{\omega^4 - 4(\alpha_1\bar{\omega}_1)^2}}{2}$$

therefore

$$\alpha_1 = \frac{(\alpha_1\bar{\omega}_1)}{\bar{\omega}_1} = \frac{(\alpha_1\bar{\omega}_1)}{\sqrt{\dfrac{\omega_1^2 + \sqrt{\omega_1^4 - 4(\alpha_1\bar{\omega}_1)^2}}{2}}}$$

From Appendix C,

$$\Delta = \frac{2\pi\alpha_1}{\bar{\omega}_1} = \frac{2\pi(\alpha_1\bar{\omega}_1)}{\bar{\omega}_1^2} = \frac{4\pi(\alpha_1\bar{\omega}_1)}{\omega^2 + \sqrt{\omega^4 - 4(\alpha_1\bar{\omega}_1)^2}}$$

Substituting value of $(\alpha_1\bar{\omega}_1)$ from equation (33), the logarithmic decrement can be expressed as:

$$\Delta = \frac{4\pi \cdot \frac{\pi^6}{2a^6} \cdot \frac{c}{\rho} \cdot \frac{G_d}{G_{xz}^2} \frac{E^2}{(1-\nu^2)^2\left(c+\frac{f+f'}{2}\right)^2} \cdot \frac{I_T^2}{(1+s)^2}}{\frac{\pi^4 E}{a^4(1-\nu^2)\rho}\left(\frac{I_T}{1+s}+I_F\right) + \sqrt{\frac{\pi^8E^2}{a^8(1-\nu^2)^2\rho^2}\left(\frac{I_T}{1+s}+I_F\right)^2 - \frac{\pi^{12}}{a^{12}} \cdot \frac{c^2}{\rho^2} \cdot \frac{G_d}{G_{xz}^4} \cdot \frac{E^4}{(1-\nu^2)^4\left(c+\frac{f+f'}{2}\right)^4} \cdot \frac{I_T^4}{(1+s)^4}}}$$

$$= \frac{\frac{2\pi^3}{a^2} \cdot \frac{c}{(1-\nu^2)} \cdot \frac{G_d}{G_{xz}^2} \cdot \frac{E}{\left(c+\frac{f+f'}{2}\right)^2} \cdot \frac{I_T^2}{(1+s)^2}}{\left(\frac{I_T}{1+s}+I_F\right) + \sqrt{\left(\frac{I_T}{1+s}+I_F\right)^2 - \frac{\pi^4c^2}{a^4} \cdot \frac{G_d^4}{G_{xz}^4} \cdot \frac{E^2}{\left(c+\frac{f+f'}{2}\right)^4} \cdot \frac{I_T^4}{(1+s)^4} \cdot \frac{1}{(1-\nu^2)^2}}}$$

As the thickness of the facings is very small, $I_F \approx 0$. Therefore, simplifying the above equation

$$\Delta = \frac{2\pi^3 c \, \frac{G_d}{G_{xz}} \, \frac{E f f'}{[a^2(f+f')G_{xz}(1-\nu^2) + \pi^2 c f f' E]}}{1 + \sqrt{1 - \pi^4 c^2 \cdot \frac{G_d^2}{G_{xz}^2} \cdot \frac{E^2 f^2 f'^2}{[a^2(f+f')G_{xz}(1-\nu^2) + \pi^2 c f f' E]^2}}}$$

For equal thickness of the facings, $f' = f$, so that

$$\Delta = \frac{2\pi \frac{G_d}{G_{xz}} \cdot \frac{1}{\left[\frac{1}{\pi^2} \cdot \frac{G_{xz}}{E} \cdot \frac{a^2}{fc}(1-\nu^2) + 1\right]}}{1 + \sqrt{1 - \frac{G_d^2}{G_{xz}^2} \cdot \frac{1}{\left[\frac{2}{\pi^2} \cdot \frac{G_{xz}}{E} \cdot \frac{a^2}{fc}(1-\nu^2) + 1\right]^2}}}$$

Since $G_{xz} >> G_d$

$$\Delta \approx \pi \frac{G_d}{G_{xz}} \cdot \frac{1}{\left[\frac{1}{\pi^2} \cdot \frac{G_{xz}}{E} \cdot \frac{a^2}{fc}(1-\nu^2) + 1\right]}$$

Looking as this equation, it can be predicted that:

1. Damping is independent of overall density of the beam
2. Damping increases if:
   a. a^2/fc decreases

b. G_{xz}/E decreases

c. G_d/G_{xz} increases.

However, since the factor $G_{xz}/E << 1$, one finds that the change in a, f and c has very small effect on the overall damping. The damping is principally dependent on G_d/G_{xz} ratio, which, if very sensitive to change in frequency, causes considerable change in the value of logarithmic decrement.

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